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The Shadow of a Straight Edge

E. T. Hanson

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THE SHADOW OF A STRAIGHT EDGE*

By E. T. HANSON, B.A.

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CONTENTS

Introduction

In his anniversary address on 30 November 1935, the President of the Royal Society paid a tribute to the work of Hector Munro Macdonald. In the course of his address he made particular reference to Macdonald's solution of the difficult, and analytically very attractive, problem of diffraction by a prism, perhaps better known as the Wedge Problem.

A few years ago the author of the present paper, who was much impressed by the work of Macdonald and had the privilege of corresponding with him, made some notes upon this problem and upon an associated problem of particular interest. Arising from these notes, which were recalled upon reading the President's address, certain ideas, which would appear to be of a fundamental nature, have been developed to form the subject of the following communication.

The problem of the half-plane had been previously treated by two distinguished mathematicians, but the first complete solution of the wedge problem was given by Macdonald. The two mathematicians were Poincaré and Sommerfeld. Subsequently to the publication of Macdonald's work, Bromwich published a paper which gives a slightly more general treatment of the problem. References are given at the end of the paper.

The importance of the problem, from the physical point of view, lies in the analytical verification of Fresnel's theory. In this the authors quoted have been brilliantly successful, but none of their papers is of recent date and the problem is still full of interest and difficulty.

The problem of a half-plane which is not infinitely thin was dealt with by the author in 1930.

* The author is indebted to the Admiralty for permission to publish this paper.

The methods of MACDONALD and BROMWICH are developments of the method of images, as distinct from the method of POINCARÉ and that of SOMMERFELD, who for the half-plane introduced functions of period 4π.

It is remarkable, however, that no writer has given due consideration to the geometrical shadow, which is the most fundamental feature of the problem. If one proceeds far along the shadow from the diffracting edge, a point is reached where the phenomena on either side are independent of the properties of the wedge so long as it does not transmit the waves falling upon it.

The full significance of the geometrical shadow is explained in the analysis which occupies the first section of the paper.

The second section deals with some points in the analysis when the incident wave is plane.

The third section considers the case of total reflexion in the presence of diffraction by a straight edge, in which the geometrical shadow plays an important part. Notwithstanding its bearing upon the interesting problem of the superficial wave in total reflexion, there does not appear to have been any previous treatment. The superficial wave was considered by STOKES and KELVIN, who called it the "clinging" wave, and others, and also by the author in 1926.

In the fourth section the result of finite thickness in a half-plane is investigated, showing that the effect upon the shadow at great distances is negligible.

In the fifth section the effect upon the shadow caused by bringing the source close to the edge is considered.

The general method of solution described in the first and second sections is not dependent upon the polarization, the plane of polarization being usually defined as the plane containing the magnetic force while the electric force is perpendicular to it. Throughout the paper, for brevity, it is assumed that when the incident wave is plane the electric force is parallel to the edge of the screen or wedge, while, in the event of the source being a Hertzian oscillator, the axis of the oscillator is parallel to the edge of the screen.

In the sections so far enumerated the incident wave possesses a geometrical shadow. This is naturally the case of greatest importance on account of its connexion with the fundamental theories of FRESNEL and STOKES. But to complete the wedge problem all cases must be included. This extension is straightforward when the wedge is perfectly reflecting. When, however, the wedge is perfectly absorbing difficulties arise when the incident wave does not possess a geometrical shadow. The difficulties exist in MACDONALD's theory of absorbing bodies which will be discussed in the final section. In the first five sections these difficulties do not affect the argument, but nevertheless a statement of MACDONALD's theory is most conveniently given here.

"A perfectly absorbing body may be regarded as a body which is incapable of supporting either electric or magnetic force. Hence if C is the electric current distribution on the surface of the body when it is supposed to be perfectly conducting, and

C' is the magnetic current distribution on the surface of the body when it is supposed to be incapable of supporting magnetic force, the superposition of these two distributions gives the electric and magnetic current distributions on the surface of the body when it is perfectly absorbing."

Suppose that the electric force E_z is parallel to the edge of the screen. Then the solution for a perfectly conducting screen may be put in the form

$$E_z = f\left\{\cos\frac{\pi}{\omega}(\theta-\theta_0)\right\} - f\left\{\cos\frac{\pi}{\omega}(\theta+\theta_0)\right\}.$$

This expression satisfies the boundary conditions because it vanishes over the two faces of the screen corresponding respectively to $\theta = 0$ and $\theta = \omega$. If the screen is incapable of supporting magnetic force, then $\partial E_z/\partial\theta$ must vanish over the faces of the screen. Under these circumstances the solution is

$$E_z = f\left\{\cos\frac{\pi}{\omega}(\theta-\theta_0)\right\} + f\left\{\cos\frac{\pi}{\omega}(\theta+\theta_0)\right\}.$$

Hence MACDONALD's theory implies that the solution when the screen is perfectly absorbing is

$$E_z = f\left\{\cos\frac{\pi}{\omega}(\theta-\theta_0)\right\}.$$

The angle ω of the wedge or screen is the external angle. In the case of the half-plane $\omega = 2\pi$. All wedges may be considered by giving ω all values between 0 and 2π.

1—THE POINT SOURCE

The problem before us in this section is one in which an origin of co-ordinates is taken upon the diffracting edge and a source at a finite distance from the origin emitting a system of waves. Starting with a solution of period 2π it is proposed to derive a solution of the wave equation of period 2ω and to consider the limits to which the components of the latter solution tend as the source is brought closer and closer to the origin.

In the first place a general theorem is required. In terms of cylindrical co-ordinates the wave equation is

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=0.$$

We commence with a solution of the wave equation of period 2π in θ, which is assumed to be expressible in the form

$$F\{r, \cos(\theta-\theta_0), z, t\}. \tag{1}$$

From (1) may be derived the solution

$$\int_a^b F\{r, \cos(u+\epsilon), z, t\} f(u)\, du, \tag{2}$$

in which ϵ has been written for $\theta-\theta_0$, a and b are arbitrary constants, and u is independent of r, θ, z, and t.

Let $u+\epsilon = v$. Then (2) may be written in the form

$$\int_{a+\epsilon}^{b+\epsilon} F\{r, \cos v, z, t\} f(v-\epsilon)\, dv,$$

which is equivalent to

$$\left[\int_a^b + \int_b^{b+\epsilon} - \int_a^{a+\epsilon}\right] F\{r, \cos v, z, t\} f(v-\epsilon)\, dv.$$

If the function f is periodic with a period 2π, and if $b = \pi$ and $a = -\pi$, then

$$\int_b^{b+\epsilon} = \int_a^{a+\epsilon}.$$

Hence, if f is periodic with a period 2π,

$$\int_{-\pi}^{\pi} F\{r, \cos v, z, t\} f(v-\epsilon)\, dv \tag{3}$$

is a solution.

The variable v thus introduced may be complex and when this is the case it is convenient to denote it by ζ, where

$$\zeta = \xi + i\eta.$$

Consider, therefore, the expression

$$\int F\{r, \cos \zeta, z, t\} f(\zeta-\epsilon)\, d\zeta. \tag{4}$$

If both the upper and the lower limits of a contour in the ζ-plane are infinite and if the integrand of (4) vanishes at each of these limits, then, whatever may be the period of the function f, the integral (4) taken round the contour between these limits is a solution of the wave equation.

It is clear that $f(\zeta-\epsilon)$ may be replaced by $f_0(\zeta+\epsilon)$, where f_0 is some other function consistent with the above conditions. Hence, upon writing

$$f(\zeta-\epsilon) + f_0(\zeta+\epsilon) = g(\zeta),$$

$$\int F\{r, \cos \zeta, z, t\}\, g(\zeta)\, d\zeta \tag{5}$$

is a solution.

In order to determine a suitable contour, the solution when no wedge is present must be known, so that a form can be given to the function F. Also an expansion of period 2ω must be adopted for the function $g(\zeta)$.

The function F is chosen so that it represents the solution of the wave equation for waves diverging from a point source S (fig. 1). The section of the wedge is AOB in the figure, O being the origin of co-ordinates. The distance of a point P from S is denoted by R. The co-ordinates of S are r_0, θ_0, 0, and the co-ordinates of P are r, θ, z. The plane containing OQ the continuation of SO, and bounded by the edge of the wedge is the geometrical shadow.

The required solution is given by

$$\frac{F_0(ct-R)}{R},$$

and
$$R^2 = r^2+r_0^2-2rr_0\cos(\theta-\theta_0)+z^2.$$

If the form of the function F_0 is arbitrary, a very general solution of the problem may be obtained. It will, however, be assumed, as was done by MACDONALD, that the source is sending out a train of waves continuously so that a steady state has been reached. Thus it is sufficient to consider one of its harmonic constituents. The function $F_0(ct-R)$ is accordingly replaced by

$$Ae^{ik(ct-R)},$$

where $\lambda, = 2\pi/k$, is the wave-length of the particular constituent. This function may be used to develop the theory of the Hertzian oscillator.

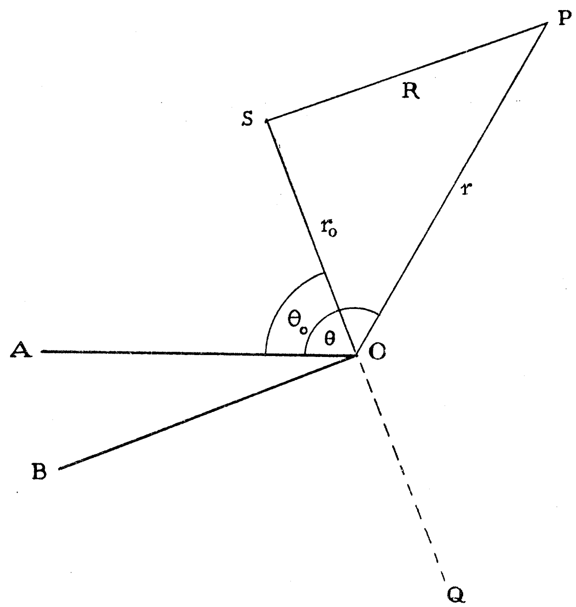

FIG. 1

It is convenient to commence by taking $\lambda/2\pi$ as the unit of length, so that $k = 1$, for k can be restored at the end of the analysis. Writing, therefore,

$$\rho^2 = r^2+r_0^2-2rr_0\cos\zeta+z^2,$$

we have to put
$$F\{r, \cos\zeta, z, t\} = A\frac{e^{-i\rho}}{\rho}e^{ict}. \tag{6}$$

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The function to be discussed then reduces to

$$\int \frac{e^{-i\rho}}{\rho} g(\zeta)\, d\zeta, \tag{7}$$

and it is necessary to consider this function as a function of ζ.

In the ζ-plane (fig. 2) there is a branch point of the integrand of (7) at P, whose co-ordinates are

$$\xi = 0 \quad \text{and} \quad \eta = \cosh^{-1}\left\{\frac{r^2+r_0^2+z^2}{2rr_0}\right\}.$$

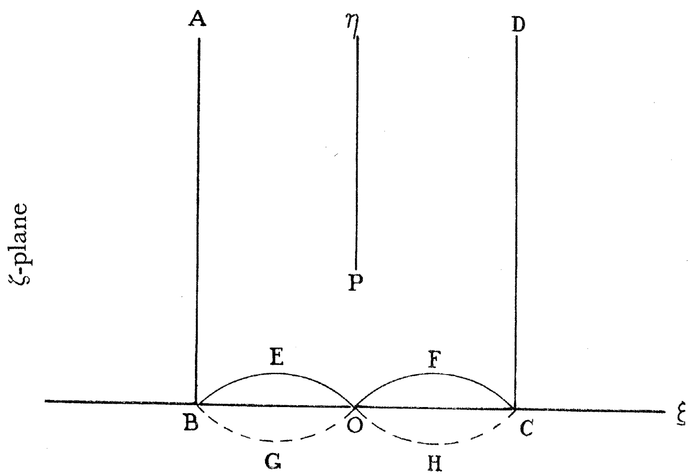

Fig. 2

The corresponding branch line is taken to coincide with the imaginary axis from P to $+\infty$. There is another branch line, which is the image of this in the axis of ξ, extending from the image of P along the imaginary axis to $-\infty$. The branch lines thus chosen enable us to determine a contour at every point of which the integrand is single-valued. When the period of $g(\zeta)$ is 2π the path of integration is one passing from B to C, where B is the point $(-\pi, 0)$, and C is the point $(+\pi, 0)$. This path may be represented provisionally by $BEOFC$, and it must pass through O because points infinitely close to the source correspond to positions of P (fig. 2) and its image infinitely close to O, and the path must not cut the branch lines. When the period of $g(\zeta)$ is 2ω the path may be extended vertically from B to A and from C to D, where A and D are at infinity. In this way the integral taken along the path $ABEOFCD$ is a solution for any value of ω, which reduces to the integral taken along the path $BEOFC$ when $\omega = \pi$. The portion $AB+CD$ of the path is thus determined, for it will presently be clear why it is taken entirely in one-half of the ζ-plane. But the remainder $BEOFC$ is not determined, and there are no means at this stage available for its determination.

The integral taken along the path $AB+CD$ cannot be assumed to be a solution, but it is possible to make it so by suitably restricting the function $g(\zeta)$.

We now require a definition of diffraction which is both simple and comprehensive, and we shall assume that the integral taken along the path $AB+CD$ satisfies this definition because it vanishes when the period of $g(\zeta)$ is 2π, this period corresponding to the absence of the wedge. The simplest and most fundamental property possessed by a large and important class of diffraction phenomena depends upon the difference between ultimate effects on either side of the geometrical shadow. In the case of a source of unit strength this difference amounts to $|e^{-iR}/R|$, since the ultimate effects inside the shadow vanish. We may assume then that under certain circumstances this property is possessed by the complete solution of our problem.

Basing the definition of diffraction upon this property we make the assumption that circumstances exist under which the integral taken along the path $AB+CD$ tends to the value $(1+M)\,e^{-iR}/R$ outside the geometrical shadow and to the value Me^{-iR}/R inside. The circumstances under which these conditions exist with reference to the integral under consideration have to be determined, and it will be shown that we are then led to the solution of the problem by means of FOURIER's theorem. That there must be a close connexion between this theorem and diffraction is almost obvious, but the actual analytical relationship is not so obvious. If under the circumstances we can make the integral satisfy the conditions outside and inside the shadow by means of the function $g(\zeta)$, we must then endeavour to specify the path $BEOFC$ and the constant M so as to satisfy the conditions of the problem, and thus finally the solution we are seeking will be the integral taken round the path $ABEOFCD$.

We proceed to show that $g(\zeta)$ is uniquely determined by these conditions. We have for this purpose to consider that part of (7) which is given by

$$\left[\int_A^B+\int_C^D\right]\frac{e^{-i\rho}}{\rho}\{f(\zeta-\epsilon)+f_0(\zeta+\epsilon)\}\,d\zeta. \tag{8}$$

Since the path of integration has been taken entirely in the upper half of the ζ-plane, and since f and f_0 are assumed to be functions of period 2ω, $g(\zeta)$ may be expanded in the form of the trigonometrical series

$$\sum_{n=0}^{n=\infty}\left[\tfrac{1}{2}a_n e^{i\frac{n\pi}{\omega}(\zeta-\epsilon)}+\tfrac{1}{2}b_n e^{i\frac{n\pi}{\omega}(\zeta+\epsilon)}\right], \tag{9}$$

which is equivalent to
$$\sum_{n=0}^{n=\infty} e^{i\frac{n\pi}{\omega}\zeta}C_n,$$

where
$$C_n = A_n\cos\frac{n\pi}{\omega}\epsilon+B_n\sin\frac{n\pi}{\omega}\epsilon,$$

and A_n and B_n are linear functions of a_n and b_n. This is permissible since at all points of the path the real part of $in\pi\zeta/\omega$ is negative. Every term of the series vanishes when $\eta=\infty$, except the term in which $n=0$. This term may, however, be included, for $e^{-i\rho}/\rho$ vanishes at the infinite points of the path. Thus, denoting (8) by ϕ, we may write

$$\phi = \left[\int_A^B + \int_C^D\right] \frac{e^{-i\rho}}{\rho} \sum_{n=0}^{n=\infty} C_n e^{i\frac{n\pi}{\omega}\zeta} d\zeta, \tag{10}$$

provided that the series is convergent.

Now along AB $\zeta = -\pi + i\eta$ and along CD $\zeta = +\pi + i\eta$. Hence

$$\phi = \sum_{n=0}^{n=\infty} i\left(e^{i\frac{n\pi^2}{\omega}} - e^{-i\frac{n\pi^2}{\omega}}\right) C_n \int_0^\infty \frac{e^{-i\rho_1}}{\rho_1} e^{-\frac{n\pi}{\omega}\eta} d\eta,$$

where $$\rho_1^2 = r^2 + r_0^2 + 2rr_0 \cosh \eta + z^2$$

and is positive at all points of the path. Thus

$$\phi = -\sum_{n=0}^{n=\infty} 2 \sin \frac{n\pi^2}{\omega} C_n \int_0^\infty \frac{e^{-i\rho_1}}{\rho_1} e^{-\frac{n\pi}{\omega}\eta} d\eta, \tag{11}$$

and obviously vanishes when $\omega = \pi$.

In (11) it is in the first place necessary to consider the integral

$$D_n = \int_0^\infty \frac{e^{-i\rho_1}}{\rho_1} e^{-\frac{n\pi}{\omega}\eta} d\eta.$$

Now $$\int_0^\infty e^{-\left(a_1{}^2x^2 + \frac{b^2}{x^2}\right)} dx = \frac{\pi^{\frac{1}{2}}}{2a_1} e^{-2a_1 b},$$

a result which holds for real positive values of a_1^2 and b^2, or for purely imaginary values. If we write, therefore,

$$a_1 = i^{\frac{1}{2}}\left(\frac{r^2 + r_0^2 + z^2}{2rr_0} + \cosh \eta\right)^{\frac{1}{2}}$$

and $$b = \tfrac{1}{2} i^{\frac{1}{2}} (2rr_0)^{\frac{1}{2}},$$

then $$2a_1 b = i\rho_1$$

and $$2a_1 = \frac{i^{\frac{1}{2}}}{(\frac{1}{2}rr_0)^{\frac{1}{2}}} \rho_1.$$

Hence $$(\tfrac{1}{2}rr_0)^{\frac{1}{2}} \left(\frac{\pi}{i}\right)^{\frac{1}{2}} \frac{e^{-i\rho_1}}{\rho_1} = \int_0^\infty e^{-i(\gamma + \cosh \eta) x^2 - i\frac{rr_0}{2x^2}} dx,$$

where $$\gamma = \frac{r^2 + r_0^2 + z^2}{2rr_0}.$$

Thus $$D_n = \left(\frac{2i}{\pi rr_0}\right)^{\frac{1}{2}} \int_0^\infty \int_0^\infty e^{-i(\gamma + \cosh \eta) x^2 - i\frac{rr_0}{2x^2} - \frac{n\pi}{\omega}\eta} dx\, d\eta.$$

Carrying out the integration with respect to η we have to consider

$$E_n = \int_0^\infty e^{-ix^2\cosh\eta - \frac{n\pi}{\omega}\eta}\,d\eta.$$

Let $$ix^2\cosh\eta + \frac{n\pi}{\omega}\eta = \sigma.$$

Then $$E_n = -\left|e^{-\sigma}\frac{d\eta}{d\sigma}\right| - \left|e^{-\sigma}\frac{d^2\eta}{d\sigma^2}\right| - \dots.$$

If we put $$\frac{d\eta}{d\sigma} = \frac{1}{p}$$

so that $$p = ix^2\sinh\eta + \frac{n\pi}{\omega},$$

then $$\frac{d^2\eta}{d\sigma^2} = -\frac{1}{p^3}\frac{dp}{d\eta},$$

$$\frac{d^3\eta}{d\sigma^3} = -\frac{1}{p^4}\frac{d^2p}{d\eta^2} + \frac{3}{p^5}\left(\frac{dp}{d\eta}\right)^2, \text{ etc.}$$

Hence $$E_n = \left[e^{-\sigma}\left\{\frac{d\eta}{d\sigma} + \frac{d^2\eta}{d\sigma^2} + \dots\right\}\right]_{\eta=0}$$

$$= e^{-ix^2}\left[\frac{\omega}{n\pi} - ix^2\left(\frac{\omega}{n\pi}\right)^3 - 3x^4\left(\frac{\omega}{n\pi}\right)^5 - \dots\right].$$

Thus $$D_n = \left(\frac{2i}{\pi r r_0}\right)^{\frac{1}{2}}\int_0^\infty e^{-i(\gamma+1)x^2 - i\frac{rr_0}{2x^2}}\left[\frac{\omega}{n\pi} - ix^2\left(\frac{\omega}{n\pi}\right)^3 - \dots\right]dx.$$

Let us now write $$i^{\frac{1}{2}}(\gamma+1)^{\frac{1}{2}} = a$$

and $$P = \int_0^\infty e^{-a^2x^2 - \frac{b^2}{x^2}}\,dx = \frac{\pi^{\frac{1}{2}}}{2a}e^{-2ab}.$$

Then, writing $$P_1 = \int_0^\infty e^{-a^2x^2 - \frac{b^2}{x^2}}x^2\,dx,$$

we have $$P_1 = -\frac{1}{2a}\frac{dP}{da}$$

$$= \frac{\pi^{\frac{1}{2}}}{2a}\frac{b}{a}e^{-2ab},$$

retaining only the lowest power of $1/2a$. Likewise, writing

$$P_2 = \int_0^\infty e^{-a^2x^2 - \frac{b^2}{x^2}}x^4\,dx,$$

we have $$P_2 = -\frac{1}{2a}\frac{dP_1}{da}$$

$$= \frac{\pi^{\frac{1}{2}}}{2a}\left(\frac{b}{a}\right)^2 e^{-2ab},$$

retaining again only the lowest power of $1/2a$. Now

$$\frac{b}{a} = \frac{(\frac{1}{2}rr_0)^{\frac{1}{2}}}{\left\{\frac{r^2+r_0^2+z^2}{2rr_0}+1\right\}^{\frac{1}{2}}}.$$

Hence if the source comes very close to the origin so that r_0 becomes very small, both $1/2a$ tends to vanish and b/a tends to vanish. This will also be the case if z becomes very large. Under these circumstances powers of $\omega/n\pi$ above the first in D_n may be neglected and in the limit, as $r_0 \to$ zero,

$$D_n = \left(\frac{2i}{\pi r r_0}\right)^{\frac{1}{2}} \frac{\pi^{\frac{1}{2}}}{2a} e^{-2ab} \frac{\omega}{n\pi}$$

$$= \frac{1}{R_0} e^{-iR_0} \frac{\omega}{n\pi},$$

where $$R_0^2 = r^2+r_0^2+z^2+2rr_0.$$

When $n = 0$, $C_n = A_0$, and the factor $\sin n\pi^2/\omega$ in the first term of the series for ϕ expressed by (11) vanishes whatever the value of ω. Hence in the limit (11) reduces to

$$\phi = -\frac{e^{-iR_0}}{R_0} \frac{2\omega}{\pi} \sum_{n=1}^{n=\infty} \frac{1}{n} \sin \frac{n\pi^2}{\omega} C_n. \tag{12}$$

Let us seek to satisfy the definition of diffraction with the series obtained by putting

$$C_n = A_n \cos \frac{n\pi}{\omega} \epsilon,$$

so that in the first place we have to consider

$$\phi_1 = \frac{e^{-iR_0}}{R_0} \left[-\frac{2\omega}{\pi} \sum_{n=1}^{n=\infty} \frac{1}{n} A_n \sin \frac{n\pi^2}{\omega} \cos \frac{n\pi}{\omega} \epsilon \right]. \tag{13}$$

The expression within brackets in (13) is a Fourier's series of cosines, but the first term, corresponding to $n = 0$, is absent. It represents the same function of ϵ in the interval 0 to ω as it does in the interval 0 to $-\omega$. The condition to be satisfied is that the expression within brackets in (13), which may be denoted by S, should have the value $1+M$ outside the geometrical shadow and the value M inside.

The interval within which S must be capable of representing an arbitrary function is the external angle of the wedge. For if in the extreme case the source were situated on one face of the wedge, S would require to be capable of representing an arbitrary function in that interval. Hence ω is the external angle of the wedge.

Now if S has the value $1+M$ when

$$0 < \epsilon < \pi,$$

and the value M when $$\pi < \epsilon < \omega,$$

S may be expanded in the form

$$S = M+\frac{\pi}{\omega}+\frac{2}{\pi}\sum_{n=1}^{n=\infty}\frac{1}{n}\sin\frac{n\pi^2}{\omega}\cos\frac{n\pi}{\omega}\epsilon. \tag{14}$$

Comparing coefficients of (13) and (14) we must put

$$M = -\pi/\omega \quad\text{and}\quad \omega A_n = -1.$$

Thus the cosine series determines M uniquely.

At first sight it might be concluded that, if the sine series in C_n were added, any arbitrary constant value might be assigned to M, for such a series may be found which possesses any constant value in the interval 0 to ω. This, however, is not permissible, for the sine series is discontinuous when $\epsilon = 0$. Hence

$$g(\zeta) = A_0-\frac{1}{\omega}\sum_{n=1}^{n=\infty} e^{i\frac{n\pi}{\omega}\zeta}\cos\frac{n\pi}{\omega}\epsilon.$$

Writing $t = e^{+\frac{i\pi(\zeta+\epsilon)}{\omega}}$ and $u = e^{+\frac{i\pi(\zeta-\epsilon)}{\omega}}$,

$$g(\zeta) = A_0-\frac{1}{2\omega}\left\{1+\frac{t}{1-t}+\frac{u}{1-u}-1\right\}$$

$$= A_0+\frac{1}{2\omega}-\frac{1}{2\omega}\frac{1-ut}{1-(u+t)+ut}$$

$$= A_0+\frac{1}{2\omega}+\frac{i}{2\omega}\frac{\sin\frac{\pi\zeta}{\omega}}{\cos\frac{\pi\zeta}{\omega}-\cos\frac{\pi\epsilon}{\omega}}.$$

Now, referring to fig. 2, suppose that AB were extended vertically downwards to infinity at D' and that DC were extended vertically downwards to infinity at A'. On account of symmetry the path $A'CHOGBD'$ may be taken as exactly equivalent to the path $ABEOFCD$, and in order that this may be the case we must put

$$A_0 = -1/2\omega.$$

It is clear, upon inspection of $g(\zeta)$, that the path $D'BGOHCA'$ is inadmissible. The integral taken along the path $AB+CD$ is then a solution, for since now

$$g(\zeta) = -g(-\zeta),$$

a path $BEOFC$ can always be found along which the integral vanishes. Hence there is obtained for ϕ the solution

$$\phi = \left[\int_A^B+\int_C^D\right]\frac{i}{2\omega}\frac{e^{-i\rho}}{\rho}\frac{\sin\frac{\pi\zeta}{\omega}}{\cos\frac{\pi\zeta}{\omega}-\cos\frac{\pi\epsilon}{\omega}}d\zeta, \tag{15}$$

which, when the source comes close to the origin, has the value

$$\left(1-\frac{\pi}{\omega}\right)\frac{e^{-iR_0}}{R_0} \tag{16}$$

outside the geometrical shadow, and the value

$$-\frac{\pi}{\omega}\frac{e^{-iR_0}}{R_0} \tag{17}$$

inside the shadow.

The function $g(\zeta)$ is thus uniquely determined by the condition that, when the source approaches very close to the origin, the value of ϕ outside the geometrical shadow differs from its value inside by e^{-iR}/R.

The remainder of the contour may now be specified. Taking it as the path *BEOFC* (fig. 2), since $g(\zeta)=-g(-\zeta)$ the integral taken along this path is the same as the integral taken round the loop *OFCHO*. Now in problems in which a geometrical shadow exists ω must be greater than π, and therefore when $\epsilon<\pi$ the loop contains only the simple pole $\zeta=\epsilon$ of the integrand and no other singularities.

Again, near the source that part of the integral contributed by the loop becomes of greater and greater importance, so that the solution satisfies the conditions near the source. It is not difficult to show that the complete solution is continuous across the geometrical shadow, and it clearly reduces to the solution for a point source in the absence of the wedge when the period is 2π. The solution obtained is applicable to an absorbing screen and is in accordance with MACDONALD's theory of absorbing bodies.

The most important case of a diffracting wedge is the half-plane, which provides an interesting verification of the theory. In the case of the half-plane

$$\omega=2\pi,$$

and therefore (15) becomes

$$\phi=\frac{i}{4\pi}\left[\int_A^B+\int_C^D\right]\frac{e^{-i\rho}}{\rho}\frac{\sin\frac{1}{2}\zeta}{\cos\frac{1}{2}\zeta-\cos\frac{1}{2}\epsilon}\,d\zeta. \tag{18}$$

Transform ζ by the substitution

$$\cos\tfrac{1}{2}\zeta=t\cos\tfrac{1}{2}\epsilon,$$

where $t=u+iv$.

In the first place let $\epsilon<\pi$, so that $\cos\frac{1}{2}\epsilon$ is positive. Thus when

$$\zeta=-\pi+i\infty,\quad t=+i\infty$$

and when $$\zeta=+\pi+i\infty,\quad t=-i\infty.$$

Hence
$$\phi = -\frac{i}{4\pi}\left[\int_{i\infty}^{0}+\int_{0}^{-i\infty}\right]\frac{e^{-i\rho}}{\rho}\frac{2dt}{t-1}$$
$$= -\frac{i}{4\pi}\int_{0}^{\infty}\frac{e^{-i\rho}}{\rho}\left\{-\frac{2i}{it-1}+\frac{2i}{it+1}\right\}dt$$
$$= \frac{1}{\pi}\int_{0}^{\infty}\frac{e^{-i\rho}}{\rho}\frac{1}{t^2+1}\,dt.$$

Substituting for $\cos\zeta$ in ρ we have
$$\rho^2 = R^2+4rr_0\cos^2\tfrac{1}{2}\epsilon(t^2+1).$$

Writing
$$\alpha = 2(rr_0)^{\frac{1}{2}}\,|\cos\tfrac{1}{2}\epsilon\,|,$$

it can be shown that the above expression for ϕ may be written in the form
$$\phi = \frac{i}{\pi}\int_{\alpha}^{\infty}\frac{K_1\{i\surd(R^2+w^2)\}}{\surd(R^2+w^2)}\,dw,$$

where K_1 is the modified Bessel function of the second kind and first order. Finally, substituting
$$w = R\sinh\psi,$$
$$\phi = \frac{i}{\pi}\int_{\psi_0}^{\infty}K_1(iR\cosh\psi)\,d\psi, \tag{19}$$

where
$$\sinh\psi_0 = \frac{2}{R}(rr_0)^{\frac{1}{2}}\,|\cos\tfrac{1}{2}\epsilon\,|.$$

In like manner when $\cos\frac{1}{2}\epsilon$ is negative, that is when $\epsilon>\pi$,
$$\phi = -\frac{i}{\pi}\int_{\psi_0}^{\infty}K_1(iR\cosh\psi)\,d\psi, \tag{20}$$

where
$$\sinh\psi_0 = \frac{2}{R}(rr_0)^{\frac{1}{2}}\,|\cos\tfrac{1}{2}\epsilon\,|.$$

Now the contribution of the loop to points outside the geometrical shadow is $-e^{-iR}/R$, which is equal to
$$-\frac{i}{\pi}\int_{-\infty}^{+\infty}K_1(iR\cosh\psi)\,d\psi.$$

Hence for the complete solution outside the geometrical shadow we have
$$-\frac{e^{-iR}}{R}+\frac{i}{\pi}\int_{\psi_0}^{\infty}K_1(iR\cosh\psi)\,d\psi = -\frac{i}{\pi}\int_{-\infty}^{\psi_0}K_1(iR\cosh\psi)\,d\psi.$$

The complete solution inside the shadow is given by (20), which is equal to
$$-\frac{i}{\pi}\int_{-\infty}^{-\psi_0}K_1(iR\cosh\psi)\,d\psi.$$

We may therefore write as the required solution

$$\chi = -\frac{i}{\pi}\int_{-\infty}^{\psi_0} K_1(iR\cosh\psi)\,d\psi, \tag{21}$$

where
$$\sinh\psi_0 = \frac{2}{R}(rr_0)^{\frac{1}{2}}\cos\tfrac{1}{2}\epsilon,$$

for the integral is now continuous with ϵ. It will be seen that in the expression for the source with which we started we have to put $A = -1$.

If on restoring the factor k we substitute kR for R, then corresponding to the source $+e^{-ikR}/R$ we have the solution

$$\chi = +\frac{ik}{\pi}\int_{-\infty}^{\psi_0} K_1(ikR\cosh\psi)\,d\psi. \tag{22}$$

The foregoing analysis shows that the integral taken along the path $AB+CD$ has the effect of making the solution continuous across the geometrical shadow.

The expression (22) was obtained by MACDONALD in an entirely different manner.

2—THE INCIDENT PLANE WAVE

When the incident wave is plane the solution for a half-plane can be derived from (22) by supposing that the source moves off to infinity. In the case of the plane wave, however, whose front is parallel to the diffracting edge of a half-plane, the problem is the simplest in the theory and can be treated by a special method. The method involves the use of the parabolic substitution and leads to a partial differential equation which it is worth while to consider. LAMB has used this substitution in his "Hydrodynamics" in a rather different manner, and he has discussed only the case of perpendicular incidence. The wave equation is in two dimensions and may be written in the form

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{23}$$

Let the incident wave be denoted by

$$e^{ik\{r\cos(\theta-\theta_0)+ct\}},$$

and let us write
$$\psi = e^{ik\{r\cos(\theta-\theta_0)+ct\}}\phi.$$

Then ϕ must satisfy the equation

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+2ik\left[\frac{\partial\phi}{\partial r}\cos(\theta-\theta_0)-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\sin(\theta-\theta_0)\right] = 0.$$

If we transform this equation by the parabolic substitution

$$x = (2ikr)^{\frac{1}{2}}\cos\tfrac{1}{2}(\theta-\theta_0),$$

$$y = (2ikr)^{\frac{1}{2}}\sin\tfrac{1}{2}(\theta-\theta_0),$$

we obtain a simple partial differential equation which possesses elementary solutions. The equation is

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+2x\frac{\partial\phi}{\partial x}-2y\frac{\partial\phi}{\partial y}=0.$$

Let

$$\phi = XY,$$

where X is a function of x only and Y is a function of y only. Then X and Y satisfy the equations

$$\frac{d^2X}{dx^2}+2x\frac{dX}{dx}-2nX=0 \tag{24}$$

and

$$\frac{d^2Y}{dy^2}-2y\frac{dY}{dy}+2nY=0. \tag{25}$$

We shall consider those solutions in which n may possess all integral values including zero. The independent solutions of (24) are easily found to be

$$X_{n0}=a_0\left[1+\frac{2n}{2!}x^2+\frac{2^2n(n-2)}{4!}x^4+\frac{2^3n(n-2)(n-4)}{6!}x^6+\dots\right],$$

$$X_{n1}=a_1\left[x+\frac{2(n-1)}{3!}x^3+\frac{2^2(n-1)(n-3)}{5!}x^5+\frac{2^3(n-1)(n-3)(n-5)}{7!}x^7+\dots\right].$$

Similarly the independent solutions of (25) are

$$Y_{n0}=b_0\left[1-\frac{2n}{2!}y^2+\frac{2^2n(n-2)}{4!}y^4-\frac{2^3n(n-2)(n-4)}{6!}y^6+\dots\right],$$

$$Y_{n1}=b_1\left[y-\frac{2(n-1)}{3!}y^3+\frac{2^2(n-1)(n-3)}{5!}y^5-\frac{2^3(n-1)(n-3)(n-5)}{7!}y^7+\dots\right].$$

The components X_{n0} and Y_{n0} of the function

$$\phi = X_{n0}Y_{n0}$$

both terminate when n is even.

The components X_{n1} and Y_{n1} of the function

$$\phi = X_{n1}Y_{n1}$$

both terminate when n is odd.

All such solutions for ϕ must, however, be discarded, since they are of period 2π. Now we require a solution which changes sign as we cross the geometrical shadow and which possesses a period 4π. Thus the only permissible solutions for ϕ are

$$\phi = X_{n1}Y_{n0}.$$

Consider now an asymptotic expansion for X. Let

$$X = e^{-x^2}P.$$

Then P satisfies the equation

$$\frac{d^2P}{dx^2}-2x\frac{dP}{dx}-2(n+1)P=0.$$

This equation possesses the solution

$$P=c_0\left[x^{-(n+1)}-\frac{(n+1)(n+2)}{4}x^{-(n+3)}+\ldots\right].$$

Hence $$X=c_0e^{-x^2}\left[x^{-(n+1)}-\frac{(n+1)(n+2)}{4}x^{-(n+3)}+\ldots\right].$$

The remaining condition to be satisfied is that ϕ should vanish to the correct order at infinity, and this is the case when $n=0$. The required solution for ϕ is therefore

$$\phi=(X_{01}+X_{00})\,Y_{00},$$

X_{00} and Y_{00} being constants.

Now $$X_{01}=a_1\left\{x-\frac{2}{3!}x^3+\frac{2.2.3}{5!}x^5-\frac{2.2.2.3.5}{7!}x^7+\ldots\right\}$$

$$=a_1\int_0^x e^{-u^2}\,du.$$

Let $$X_{00}=-a_1\int_0^\infty e^{-u^2}du.$$

Then $$X_{01}+X_{00}=-a_1\int_x^\infty e^{-u^2}du,$$

which is equivalent to the asymptotic expansion.

Since $$x=(2ikr)^{\frac{1}{2}}\cos\tfrac{1}{2}(\theta-\theta_0),$$

the solution which we require is clearly

$$\psi=e^{ikr\cos(\theta-\theta_0)+ikct}\left(\frac{i}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{(2kr)^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta_0)}e^{-iv^2}\,dv. \tag{26}$$

As the upper limit $\rightarrow+\infty$ $\quad\psi\rightarrow e^{ikr\cos(\theta-\theta_0)+ikct}.$

The formula (26) is the solution for a perfectly absorbing screen and is in accordance with MACDONALD's theory of absorbing bodies as in the first section. In the final section we shall have occasion to criticize MACDONALD's theory, and it will be necessary to replace it by another theory. The two theories lead to precisely the same result, however, when the incident wave possesses a geometrical shadow. It may be remarked that the solutions of period 2π for other values of n which terminate provide a series of new solutions which are of some importance, but outside the present discussion.

3—Diffraction and total reflexion

It is proposed in this section to consider a problem which will enable us to discuss the development of the superficial wave in total reflexion.

A plane wave is incident in the direction IO in a medium M_0, which is separated from a medium M_1 by the plane boundary AOB. One-half AO of the plane boundary is covered with an infinitely thin sheet of perfectly reflecting or perfectly absorbing material. It will be assumed, in accordance with our plan, that the electric force in the wave front is parallel to the edge of the sheet, which is taken as the axis of z.

Take an origin O on the edge and let OA be the axis of x. The axis of y is supposed drawn vertically upwards. The suffixes 0, where they occur, refer to the medium M_0 and the suffixes 1 to the medium M_1.

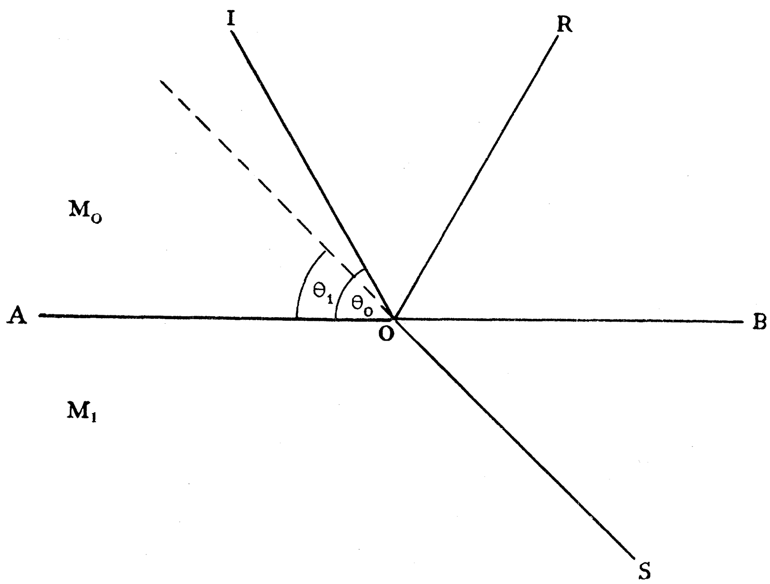

Fig. 3

Supposing in the first place that the sheet consists of perfectly absorbing material, we shall explain and consider the following expressions based upon the formula (26):

$$Ie^{ik_0V_0t},$$

$$Re^{ik_0V_0t},$$

$$Se^{ik_1V_1t},$$

where
$$I = 2e^{ik_0r\cos(\theta-\theta_0)}\left(\frac{i}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{(2k_0r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta_0)} e^{-iv^2}dv,$$

$$R = 2A_0e^{ik_0r\cos(\theta+\theta_0)}\left(\frac{i}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{-(2k_0r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta+\theta_0)} e^{-iv^2}dv,$$

$$S = 2A_1e^{ik_1r\cos(\theta-\theta_1)}\left(\frac{i}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{(2k_1r)^{\frac{1}{2}}\cos\frac{1}{2}(\theta-\theta_1)} e^{-iv^2}dv.$$

The line OR is the geometrical shadow of the wave which is partially reflected at the boundary OB between the two media. The line OS is the geometrical shadow of the wave which is refracted at the boundary OB between the two media. In the medium M_0 the reflected wave R tends to vanish when

$$\theta < \pi - \theta_0.$$

In the medium M_1 the refracted wave S tends to vanish when

$$\theta > \pi + \theta_1.$$

In the medium M_0 we shall write for the electric force

$$Z_0 = (I + R + F_0)\, e^{ik_0 V_0 t},$$

and in the medium M_1

$$Z_1 = (S + F_1)\, e^{ik_1 V_1 t}.$$

The functions F_0 and F_1 have to be determined by the conditions at the surface of separation OB, and by the condition that they must vanish at infinity.

We shall make use of the formula

$$2e^{ikr\cos\phi}\left(\frac{i}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{(2kr)^{\frac{1}{2}}\cos\frac{1}{2}\phi} e^{-iv^2}dv = e^{ikr\cos\phi} + 2\sum_{n=1}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(kr)\cos\frac{n}{2}\phi, \tag{27}$$

where $J_{\frac{n}{2}}$ is the BESSEL's function of half-integral order.

The summation is to be carried out for all positive integral odd values of n only (BATEMAN 1915).

Thus, from (27), we may write

$$I = e^{ik_0 r\cos(\theta-\theta_0)} + 2\sum_{n=1}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_0 r)\cos\frac{n}{2}(\theta-\theta_0),$$

$$R = A_0\left[e^{ik_0 r\cos(\theta+\theta_0)} + 2\sum_{n=1}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_0 r)\cos\frac{n}{2}(\theta-2\pi+\theta_0)\right],$$

$$S = A_1\left[e^{ik_1 r\cos(\theta-\theta_1)} + 2\sum_{n=1}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_1 r)\cos\frac{n}{2}(\theta-\theta_1)\right].$$

For the function F_0 we shall write

$$F_0 = 2\sum_{n=1,\,5}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_0 r)\left\{\alpha_n\cos\frac{n}{2}\theta + \beta_n\sin\frac{n}{2}\theta\right\},$$

the summation being carried out for values of $n = 1, 5, 9$, etc.

For the function F_1 we shall write

$$F_1 = 2\sum_{n=3,\,7}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_1 r)\left\{\gamma_n\cos\frac{n}{2}\theta + \delta_n\sin\frac{n}{2}\theta\right\},$$

the summation being carried out for values of $n = 3, 7, 11$, etc.

The reason for using the suffixes 1, 5, 9, etc. in F_0 and the suffixes 3, 7, 11, etc. in F_1 will be apparent presently.

Since the frequency in the two media must be the same, we must have

$$k_0 V_0 = k_1 V_1.$$

In order to satisfy the boundary conditions over the surface of separation OB between the two media when the electric force is at right angles to the plane of incidence we must have

$$\frac{\partial Z_0}{\partial y} = \frac{\partial Z_1}{\partial y} \quad \text{and} \quad Z_0 = Z_1.$$

These conditions have to be applied to the expression $I+R+F_0$ in medium M_0 and to the expression $S+F_1$ in medium M_1. From the first parts of these expressions we must have, when $\theta = \pi$,

$$k_0 \cos\theta_0 = k_1 \cos\theta_1 \quad \text{and} \quad 1+A_0 = A_1.$$

Also
$$k_0(1-A_0)\sin\theta_0 = k_1 A_1 \sin\theta_1.$$

Whence follow the well-known results, satisfied in this case at distances upon OB remote from the origin,

$$A_1 = \frac{2\sin\theta_0\cos\theta_1}{\sin(\theta_0+\theta_1)}$$

and
$$A_0 = A_1 - 1 = \frac{\sin(\theta_0-\theta_1)}{\sin(\theta_0+\theta_1)}.$$

The conditions are completely satisfied over the surface OB if in addition

$$\sum_{n=1,5}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_0 r)\left\{(1+A_0)\sin\frac{n}{2}\theta_0+\beta_n\right\} = \sum_{n=1,5}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_1 r)\left\{A_1 \sin\frac{n}{2}\theta_1\right\}, \tag{28}$$

$$\sum_{n=3,7}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_0 r)\left\{(1+A_0)\sin\frac{n}{2}\theta_0\right\} = \sum_{n=3,7}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_1 r)\left\{A_1 \sin\frac{n}{2}\theta_1+\delta_n\right\}, \tag{29}$$

$$\sum_{n=1,5}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_0 r)\, n\left\{(1-A_0)\cos\frac{n}{2}\theta_0+\alpha_n\right\} = \sum_{n=1,5}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_1 r)\, n\left\{A_1 \cos\frac{n}{2}\theta_1\right\}, \tag{30}$$

$$\sum_{n=3,7}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_0 r)\, n\left\{(1-A_0)\cos\frac{n}{2}\theta_0\right\} = \sum_{n=3,7}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_1 r)\, n\left\{A_1 \cos\frac{n}{2}\theta_1+\gamma_n\right\}. \tag{31}$$

The equations (28)–(31) have to be satisfied for all values of r.

If we had decided to consider a perfectly reflecting screen, that is a screen which cannot support electric force, we should have to subtract from I, R, and S respectively expressions I', R', and S' which differ from I, R, and S only in that the signs of θ_0 and θ_1 are changed. All the conditions in this case would be satisfied by retaining only the terms with coefficients β_n and δ_n in F_0 and F_1.

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If the screen is supposed incapable of supporting magnetic force we should have to add I', R', and S' respectively to I, R, and S. All the conditions in this case would be satisfied by retaining only the terms with coefficients α_n and γ_n in F_0 and F_1. Hence MACDONALD's theory of absorbing bodies applies. The reason for this is the existence in every wave of a geometrical shadow, and the explanation along with that of the results of the previous sections is given in the final section.

The identities (28)–(31) require for their elucidation the consideration of the equation

$$\Sigma i^{\frac{n}{2}} J_{\frac{n}{2}}(k_0 r)\, B_n = \Sigma i^{\frac{n}{2}} J_{\frac{n}{2}}(k_1 r)\, C_n. \tag{32}$$

Now $$J_{\frac{n}{2}}(\rho) = \frac{\rho^{\frac{n}{2}}}{2^{\frac{n}{2}}\Pi\left(\frac{n}{2}\right)}\left\{1-\frac{\rho^2}{2\,.\,(n+2)}+\frac{\rho^4}{2\,.\,4\,.\,(n+2)\,(n+4)}-\dots\right\},$$

and accordingly it is not difficult to write down the relations which must exist between the B's and the C's. If we write

$$(k_0/k_1)^{\frac{1}{2}} = \lambda,$$

these relations are

$$\lambda B_1 = C_1,$$

$$\lambda^3 B_3 = C_3,$$

$$\lambda^5(B_5+\tfrac{5}{2}B_1) = C_5+\tfrac{5}{2}C_1,$$

$$\lambda^7(B_7+\tfrac{7}{2}B_3) = C_7+\tfrac{7}{2}C_3,$$

$$\lambda^9\left(B_9+\tfrac{9}{2}B_5+\frac{7\,.\,9}{2\,.\,4}B_1\right) = C_9+\tfrac{9}{2}C_5+\frac{7\,.\,9}{2\,.\,4}C_1,$$

$$\lambda^{11}\left(B_{11}+\tfrac{11}{2}B_7+\frac{9\,.\,11}{2\,.\,4}B_3\right) = C_{11}+\tfrac{11}{2}C_7+\frac{9\,.\,11}{2\,.\,4}C_3,$$

and the succeeding equations may be written down by inspection.

An especially important and simple case is that in which the primary wave is incident at the critical angle. In this case $\theta_1 = 0$ and $A_0 = 1$. We shall consider this case in some detail and shall commence by using it to show why the suffixes 1, 5, 9, etc. must be selected for incorporation in the function F_0.

It may be noticed that, from our point of view, an important aspect of the surface of separation OB is that at the critical angle it becomes the geometrical shadow of the refracted wave.

The boundary conditions must, of course, be satisfied in this case, as in all cases. Along the boundary OB, since

$$I+R+F_0 = S+F_1,$$

it follows that $$\frac{d}{dr}(I+R+F_0) = \frac{d}{dr}(S+F_1).$$

Now it is clear that this would not be the case when r is very small unless we took the suffixes 1, 5, 9, etc. in F_0, because all the terms of half-integral order in medium M_1 vanish along the boundary except the terms containing δ_n. Thus if we approach the origin along a curve which at the origin is tangential to the boundary in M_1, dF_1/dr tends to vanish because r vanishes.

Likewise in medium M_0 the terms containing α_n do not vanish, but if we approach the origin along a curve which at the origin is tangential to the boundary in M_0, dF_0/dr tends to vanish because $\cos\frac{1}{2}\theta$ vanishes.

The superficial wave in medium M_1 may now be investigated without much labour. For this purpose let us return to equation (32) and the subsequent equations, in which we have to consider those possessing the suffixes 3, 7, 11, etc.

The B's being supposed known we have to find the values of the C's. For the purpose which we have in view a rough calculation shows that it is sufficient to determine the first five, that is those with the suffixes 3, 7, 11, 15, 19. In order to evaluate C_{15} and C_{19} we require the two additional equations, which can be written down by inspection,

$$\lambda^{15}\left(B_{15}+\tfrac{15}{2}B_{11}+\frac{13.15}{2.4}B_7+\frac{11.13.15}{2.4.6}B_3\right)=C_{15}+\tfrac{15}{2}C_{11}+\frac{13.15}{2.4}C_7+\frac{11.13.15}{2.4.6}C_3,$$

and

$$\lambda^{19}\left(B_{19}+\tfrac{19}{2}B_{15}+\frac{17.19}{2.4}B_{11}+\frac{15.17.19}{2.4.6}B_7+\frac{13.15.17.19}{2.4.6.8}B_3\right)$$
$$=C_{19}+\tfrac{19}{2}C_{15}+\frac{17.19}{2.4}C_{11}+\frac{15.17.19}{2.4.6}C_7+\frac{13.15.17.19}{2.4.6.8}C_3.$$

Now from equation (29), at the critical angle, we may write

$$B_n=2\sin\frac{n}{2}\theta_0 \quad \text{and} \quad C_n=\delta_n.$$

In order to obtain numerical results we shall suppose that the refractive index of medium M_0 is 1·5. In this case

$$\lambda^2=\frac{\cos\theta_1}{\cos\theta_0}=\frac{3}{2},$$

so that $\cos\theta_0=0{\cdot}666$ when $\theta_1=0$ and therefore $\theta_0=48^\circ\ 10'$. Hence we can write down the values of $\sin\frac{n}{2}\theta_0$ for $n=3, 7, 11, 15, 19$.

From the equations connecting the B's and C's it is then found that

$$\delta_3 = 1{\cdot}90, \qquad \delta_{15} = 220{\cdot}20,$$
$$\delta_7 = 8{\cdot}75, \qquad \delta_{19} = 1286{\cdot}00.$$
$$\delta_{11} = 56{\cdot}25,$$

Thus, when $\theta_1 = 0$,

$$S+F_1 = 2e^{ik_1r\cos\theta}+4\sum_{n=1}^{n=\infty} i^{\frac{n}{2}} J_{\frac{n}{2}}(k_1 r)\cos\frac{n}{2}\theta+2i^{\frac{3}{2}}[\delta_3 J_{\frac{3}{2}}(k_1 r)\sin\tfrac{3}{2}\theta-\delta_7 J_{\frac{7}{2}}(k_1 r)\sin\tfrac{7}{2}\theta$$
$$+\delta_{11}J_{\frac{11}{2}}(k_1 r)\sin\tfrac{11}{2}\theta-\delta_{15}J_{\frac{15}{2}}(k_1 r)\sin\tfrac{15}{2}\theta+\delta_{19}J_{\frac{19}{2}}(k_1 r)\sin\tfrac{19}{2}\theta],$$

the above values of the δ's being substituted when the refractive index of medium M_0 is 1·5 and of M_1 unity.

In the *British Association Reports* for 1914 and 1916 Tables of the function $S_n(x)$ have been compiled where

$$S_n(x) = \sqrt{(\tfrac{1}{2}\pi x)}\,.\,J_{n+\frac{1}{2}}(x).$$

In the 1914 Tables computations are carried out for integral values of the argument x from 1–10. In the 1916 Tables the computations are carried out for the values 1·1–1·9 of the argument. These Tables are sufficient to enable us to obtain an approximate curve of intensity of the superficial wave as it develops from the edge.

When $\theta_1 = 0$ and $\theta = \pi$, upon writing $k_1 r = x$,

$$S+F_1 = 2e^{-ix}-\frac{2i^{\frac{3}{2}}\lambda^3}{\sqrt{(\frac{1}{2}\pi x)}}[1\cdot 90 S_1(x)-8\cdot 75 S_3(x)$$
$$+56\cdot 25 S_5(x)-220\cdot 20 S_7(x)+1286\cdot 00 S_9(x)]$$
$$= 2e^{-ix}-\frac{2i^{\frac{3}{2}}\lambda^3}{\sqrt{(\frac{1}{2}\pi x)}}S(x), \text{ say.}$$

Since
$$i^{\frac{3}{2}} = -\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}},$$

we obtain finally

$$\tfrac{1}{2}(S+F_1) = \left[\cos x+\frac{\lambda^3}{\sqrt{(\pi x)}}S(x)\right]+i\left[\sin x-\frac{\lambda^3}{\sqrt{(\pi x)}}S(x)\right].$$

By means of the Tables we have computed from this expression the relative intensity of the superficial wave for the values

$$x = 1,\ 1\cdot 5,\ 2,\ 3, \text{ and } 4,$$

and from these the accompanying graph fig. 4 has been drawn.

When $x = 3$ the error is somewhat less than 2%. For smaller values of x it is still less, but for $x = 4$ it is greater. We may note, however, that if the intensity when $x = 0$ is taken as unity it should increase to 4 when $x = \infty$. Hence we must expect a rise to take place somewhere, and the calculations have been carried out far enough to indicate this rise. The most remarkable result is the reduction in intensity at about one-third of a wave length.

4—The half-plane of finite thickness

A difficulty arises in connexion with all problems of diffraction by sharp edges. The whole theory of wave motion in such problems is based upon the assumption that the displacements of the medium are everywhere small. But when an edge is sharp the solution may make the displacements at the edge infinite, so that the above assumption is broken. It is only due to the fact that these infinite displacements are confined to points very close to the edge that the theory, in its application, is but little affected.

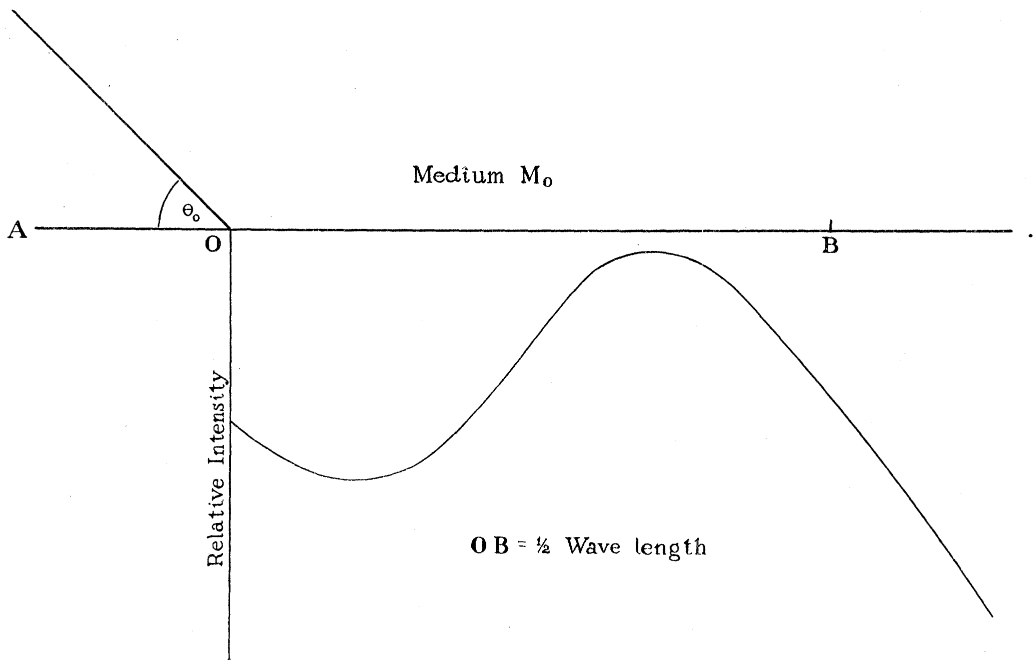

FIG. 4—Relative intensity of superficial wave at critical angle.

We shall in this instance consider a perfectly reflecting body, assuming again that the electric force in the incident wave, which is supposed to be plane, is parallel to the edge. The object of investigation is the nature of the magnetic forces in the neighbourhood from which the geometrical shadow starts. We shall take as the perfectly reflecting body a semi-infinite plane which is not thin, but whose thickness is small in comparison with the incident wave-length.

It is convenient in the first place to consider a transformation first obtained by HELMHOLTZ in connexion with a hydrodynamical problem. The transformation, which is a particular case of SCHWARTZ's transformation, is

$$u = \frac{a}{\pi}(-\chi + e^{\chi}) + \omega,$$

where $u = x+iy$, $\chi = \phi+i\psi$ and $\omega = \alpha+i\beta$, α and β being constants.

If we assume that $x = 0$ and $y = 0$ when $\phi = 0$ and $\psi = \pi$, then the transformation breaks up into

$$\left.\begin{aligned} x &= \frac{a}{\pi}(1-\phi+e^{\phi}\cos\psi), \\ y &= \frac{a}{\pi}(\pi-\psi+e^{\phi}\sin\psi). \end{aligned}\right\} \tag{33}$$

In fig. 5, AA' and BB' are the two sides of the semi-infinite plane. The side AA' corresponds to $\psi = 0$ and BB' to $\psi = 2\pi$. The edge of the plane is rounded off in the form AOB, which corresponds to $\phi = 0$. Let F be the required wave function so that

$$(\nabla^2 + k^2)\, F = 0, \tag{34}$$

where $$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2.$$

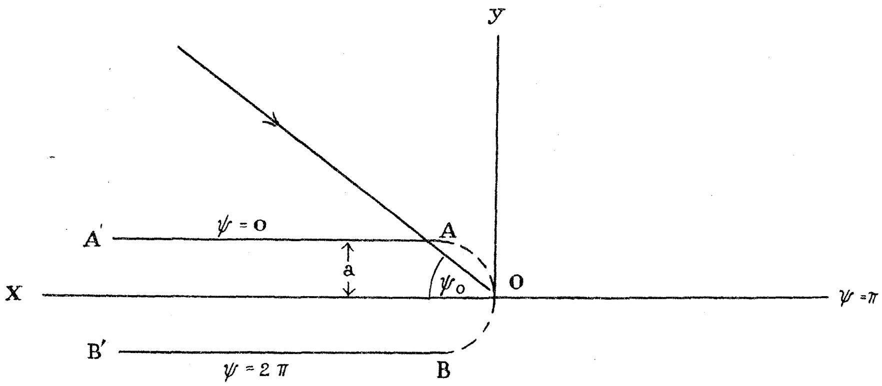

FIG. 5

Then, since $$\nabla^2 F = \left\{\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial \psi^2}\right\} \left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial x}\right)^2\right\},$$

the wave equation (34) transforms into

$$\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial \psi^2} + \frac{k^2 a^2}{\pi^2}(1 + e^{2\phi} - 2e^{\phi} \cos \psi)\, F = 0. \tag{35}$$

Putting $\frac{a}{\pi} e^{\phi} = \rho$, equation (35) becomes

$$\rho^2 \frac{\partial^2 F}{\partial \rho^2} + \rho \frac{\partial F}{\partial \rho} + \frac{\partial^2 F}{\partial \psi^2} + \left(k^2 \rho^2 + \frac{k^2 a^2}{\pi^2} - \frac{2ka}{\pi} k\rho \cos \psi\right) F = 0.$$

When ka/π is small, the case under consideration, we require a first approximation to the solution of this equation. This approximation is a solution of

$$\rho^2 \frac{\partial^2 F}{\partial \rho^2} + \rho \frac{\partial F}{\partial \rho} + \frac{\partial^2 F}{\partial \psi^2} + k^2 \rho^2 F = 0. \tag{36}$$

But this equation is precisely the same as that in cylindrical co-ordinates from which the solution of the infinitely thin semi-infinite plane is obtained. Hence when the wave is incident at an angle ψ_0, and the screen is perfectly reflecting, and the electric force in the incident wave is parallel to the edge, the approximate solution can be derived at once from § 2.

Calling this solution F_1, near the edge it reduces to

$$F_1 = \frac{1}{\sqrt{2}} e^{i\frac{1}{4}\pi} (ka)^{\frac{1}{2}} \frac{2}{\pi} e^{\frac{1}{2}\phi} \{\cos \tfrac{1}{2}(\psi - \psi_0) - \cos \tfrac{1}{2}(\psi + \psi_0)\}.$$

The function F_1 is not, however, the complete solution, for we have still to satisfy the conditions over the perfectly reflecting edge $\phi = 0$. We require, in fact, another solution of equation (36). For this purpose write

$$F = R\cos n\psi,$$

so that R must satisfy the equation

$$\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}+\left(k^2-\frac{n^2}{\rho^2}\right)R = 0.$$

Let
$$u = (k\rho)^{\frac{1}{2}}R,$$

then u must satisfy the equation

$$\frac{d^2u}{d\rho^2}-\frac{(n-\frac{1}{2})(n+\frac{1}{2})}{\rho^2}u+k^2u = 0.$$

We thus obtain for R, by putting $n = \frac{1}{2}$, the expression

$$R = Ae^{-\frac{1}{2}\phi}\,e^{-ik\frac{a}{\pi}e^{\phi}},$$

where A is a constant.

Calling the second solution of (36) F_2, we may write

$$F_2 = Ae^{-\frac{1}{2}\phi}e^{-ik\frac{a}{\pi}e^{\phi}}\{\cos\tfrac{1}{2}(\psi-\psi_0)-\cos\tfrac{1}{2}(\psi+\psi_0)\}.$$

Let the electric force parallel to the edge be denoted by E_z, then we may write

$$E_z = (F_1+F_2)\,e^{ikct}.$$

Since E_z must vanish when $\phi = 0$, we must put

$$A = -\frac{1}{\sqrt{2}}e^{i\frac{1}{4}\pi}(ka)^{\frac{1}{2}}\frac{2}{\pi}.$$

Hence near the edge

$$E_z = \frac{1}{\sqrt{2}}e^{ikct+i\frac{1}{4}\pi}(ka)^{\frac{1}{2}}\frac{8}{\pi}\sinh\tfrac{1}{2}\phi\sin\tfrac{1}{2}\psi\sin\tfrac{1}{2}\psi_0.$$

It may be observed that the approximation used in the foregoing analysis involves the vanishing of $(ka)^{\frac{3}{2}}$ and higher powers of ka in comparison with $(ka)^{\frac{1}{2}}$.

Let us write, for brevity, $\quad E_z = B\sinh\frac{1}{2}\phi\sin\frac{1}{2}\psi.$

Now the magnetic forces tangential to the curve $\phi = \text{const.}$ and normal thereto are respectively proportional at a given instant to $\partial E_z/\partial n$ and $\partial E_z/\partial s$, where ∂n is an element along the normal and ∂s an element along the tangent. But

$$\frac{\partial E_z}{\partial n} = \frac{\partial E_z}{\partial\phi}\frac{\partial\phi}{\partial n} = \tfrac{1}{2}B\cosh\tfrac{1}{2}\phi\sin\tfrac{1}{2}\psi\,\frac{\pi}{a}\,(1-2e^{\phi}\cos\psi+e^{2\phi})^{-\frac{1}{2}}$$

$$= \frac{\pi B}{4a}\quad\text{when}\quad\phi = 0.$$

Likewise $\partial E_z/\partial s = 0$ when $\phi = 0$.

Hence the magnetic force is tangential to $\phi = 0$. Let it be denoted by H_ψ. Then

$$\frac{1}{c}\frac{dH_\psi}{dt} = \frac{\partial E_z}{\partial n} = \frac{\pi B}{4a}$$

$$= \frac{1}{\sqrt{2}} e^{ikct + i\frac{1}{4}\pi} \left(\frac{k}{a}\right)^{\frac{1}{2}} 2 \sin \tfrac{1}{2}\psi_0.$$

Finally $$H_\psi = \sqrt{2} \,.\, e^{ikct - i\frac{1}{4}\pi} (ka)^{-\frac{1}{2}} \sin \tfrac{1}{2}\psi_0.$$

Thus the magnetic force is inversely proportional to $(ka)^{\frac{1}{2}}$.

From the second solution, viz. F_2, which was obtained for equation (36) it is clear that the thickness of the plane makes little difference to the phenomena at a distance from the edge. Hence the properties of the geometrical shadow at great distances are sensibly unaffected.

The case of the half-plane of finite thickness is of interest in connexion with MACDONALD's theory of absorbing bodies, and will be referred to in the final section.

5—THE SOURCE IN PROXIMITY TO THE EDGE

When the source is in proximity to the edge the effects near the shadow at great distances from the edge assume a noteworthy form. The assumptions are that r_0/R_0 and r_0/R_1 are small and that kR_0 and kR_1 are large, where R_0 refers to the incident wave and R_1 to the reflected wave.

Taking χ_0 and χ_1 as the corresponding solutions, we shall commence with the consideration of χ_0.

Now $$\chi_0 = \tfrac{1}{2}\frac{e^{-ikR_0}}{R_0} + \phi_0, \tag{37}$$

where $$\phi_0 = \frac{ik}{\pi}\int_0^{\psi_0} K_1(ikR_0 \cosh\psi)\, d\psi.$$

The integrand may be replaced by the first term of its asymptotic expansion, so that

$$\phi_0 = k^{\frac{1}{2}}(2\pi R_0)^{-\frac{1}{2}} e^{i\frac{1}{4}\pi} \int_0^{\psi_0} e^{-ikR_0\cosh\psi} (\cosh\psi)^{-\frac{1}{2}}\, d\psi.$$

Let $$(2kR_0)^{\frac{1}{2}} \sinh \tfrac{1}{2}\psi = u.$$

Then $$\cosh\psi = 1 + u^2/kR_0$$

and $$d\psi = \frac{2du}{(kR_0)^{\frac{1}{2}}(2 + u^2/kR_0)^{\frac{1}{2}}}.$$

Since ψ_0 is small for the reason that r_0/R_0 is small, u^2/kR_0 may be neglected in the non-periodic factors of the integrand. Hence

$$\phi_0 = \frac{1}{\pi^{\frac{1}{2}}R_0} e^{-ikR_0+i\frac{1}{4}\pi}\int_0^{u_0} e^{-iu^2}\,du,$$

with

$$u_0 = (2kR_0)^{\frac{1}{2}}\,.\,\tfrac{1}{2}\psi_0$$

$$= \frac{1}{R_0^{\frac{1}{2}}}(2krr_0)^{\frac{1}{2}}\cos\tfrac{1}{2}(\theta-\theta_0).$$

Thus it follows that

$$\chi_0 = \frac{1}{\pi^{\frac{1}{2}}R_0} e^{-ikR_0+i\frac{1}{4}\pi}\int_{-\infty}^{u_0} e^{-iu^2}\,du. \tag{38}$$

Now the assumptions are that r_0/R_0 is small and that kR_0 is large. Thus there is nothing to prohibit us from making kr as large as we please.

The expression (38), which is in Fresnel's form, is of interest from our point of view as providing an illustration of the circumstances under which the definition of diffraction, adopted in § 2, can be satisfied in the complete solution of our problem.

Suppose that

$$\theta-\theta_0 = \pi-\alpha,$$

so that

$$\cos\tfrac{1}{2}(\theta-\theta_0) = \sin\tfrac{1}{2}\alpha.$$

If α is small

$$u_0 = \frac{1}{2R_0^{\frac{1}{2}}}(2krr_0)^{\frac{1}{2}}\alpha.$$

Let us choose a small angle α_1. Then for a given distance r_1 we may, by making the wave-length as small as we please, that is, by making k as large as we please, make u_0 as large as we please. Hence, by making the wave-length as small as we please, χ_0 can be made to possess the value e^{-ikR_0}/R_0 on one side of the wedge of small angle $2\alpha_1$, which encloses the geometrical shadow and the value zero on the other side. The same considerations apply to the solution χ_1 for the reflected wave, which therefore need not be considered in further detail.

6—Macdonald's theory of absorbing bodies

A statement of Macdonald's theory has been made in the Introduction. There are serious difficulties in the way of its general acceptance.

We shall commence by applying it to a very simple problem, which is a particular case of the wedge problem.

Let AOB (fig. 6) be a wedge whose external angle AOB is $\frac{1}{2}\pi$. Let P_0 be the source, P_1 its image in OA, P_2 its image in OB, and P_3 the image of P_1 in $B'OB$. Let us assume that the source is a Hertzian oscillator whose axis is parallel to the edge of the wedge.

Let

$$S_n = e^{ik(ct-R_n)}/R_n.$$

If the wedge is incapable of supporting electric force, the electric current distribution C is given by the solution

$$S_0 - S_1 + S_3 - S_2.$$

If the wedge is incapable of supporting magnetic force, the magnetic current distribution C' is given by the solution

$$S_0 + S_1 + S_3 + S_2.$$

In accordance with MACDONALD's theory, if the wedge is perfectly absorbing the electric and magnetic current distributions are given by the solution

$$S_0 + S_3,$$

corresponding to the source S_0.

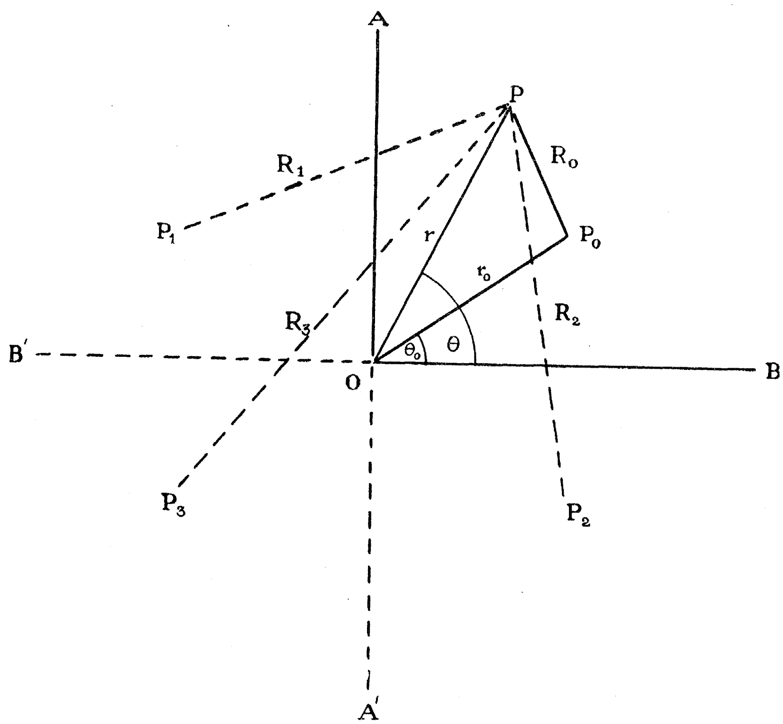

FIG. 6

It cannot reasonably be expected that such is the solution for a perfectly absorbing material. Let us, however, consider the problem from another point of view. Suppose that the wedge is incapable of supporting electric force along the face OA, and incapable of supporting magnetic force along the face OB. Then the corresponding solution is

$$S_0 - S_1 + S_2 - S_3.$$

Again, suppose that the wedge is incapable of supporting electric force along the face OB, and incapable of supporting magnetic force along the face OA. Then the corresponding solution is

$$S_0 - S_2 + S_1 - S_3.$$

For a perfectly absorbing wedge, the solution, since this alternative is quite in keeping with MACDONALD's theory, is

$$S_0 - S_3,$$

corresponding to the source S_0.

The theory appears thus to be ambiguous. Let us compare the two results.

Suppose that P_0 is moved infinitely close to OB. Then P_3 comes infinitely close to OB'. Thus MACDONALD's solution tends to the result that the face OA can support electric force but not magnetic force. On the other hand, the solution $S_0 - S_3$ in this case expresses the condition that the face OA can support magnetic force but not electric force.

Another example may be taken from the half-plane of finite thickness, which has been considered in § 4. For the purposes of the present argument it is sufficient to write down the approximate solutions near the edge.

If the surface cannot support electric force, the distribution of current is given by the solution

$$2A \sinh \tfrac{1}{2}\phi \{\cos \tfrac{1}{2}(\psi - \psi_0) - \cos \tfrac{1}{2}(\psi + \psi_0)\}.$$

If the surface cannot support magnetic force, the distribution of current is given by

$$2A \cosh \tfrac{1}{2}\phi \{\cos \tfrac{1}{2}(\psi - \psi_0) + \cos \tfrac{1}{2}(\psi + \psi_0)\}.$$

According to MACDONALD's theory, therefore, the solution for a perfectly absorbing body is

$$A\{e^{\frac{1}{2}\phi} \cos \tfrac{1}{2}(\psi - \psi_0) + e^{-\frac{1}{2}\phi} \cos \tfrac{1}{2}(\psi + \psi_0)\}.$$

But it is quite in keeping with MACDONALD's theory to argue as follows.

Referring to fig. 4, suppose that the faces $A'A$ and $B'B$ cannot support magnetic force and that the face AOB cannot support electric force. The corresponding solution is

$$2A \sinh \tfrac{1}{2}\phi \{\cos \tfrac{1}{2}(\psi - \psi_0) + \cos \tfrac{1}{2}(\psi + \psi_0)\}.$$

If, on the other hand, the faces $A'A$ and $B'B$ cannot support electric force and the face AOB cannot support magnetic force, the solution is

$$2A \cosh \tfrac{1}{2}\phi \{\cos \tfrac{1}{2}(\psi - \psi_0) - \cos \tfrac{1}{2}(\psi + \psi_0)\}.$$

If the body is perfectly absorbing, the solution turns out to be, if we accept MACDONALD's theory,

$$A\{e^{\frac{1}{2}\phi} \cos \tfrac{1}{2}(\psi - \psi_0) - e^{-\frac{1}{2}\phi} \cos \tfrac{1}{2}(\psi + \psi_0)\}.$$

A discussion of these difficulties is out of place here.

In the case of the wedge problem, however, when the incident wave possesses a geometrical shadow, the problem of absorption can be treated rather differently, and it is at this point that the reason for solving the problem by means of the properties of the geometrical shadow of the incident wave is most easily perceived.

It has been seen that the solution of the problem of the wedge is uniquely determined for the incident wave by means of these properties.

Likewise, the solution for the reflected wave is uniquely determined by means of the properties of the shadow of the reflected wave. The part scattered by diffraction is accounted for in both cases. Hence a perfectly absorbing wedge, when a geometrical shadow exists, may be defined by saying that the reflected wave disappears.

When the wedge is perfectly reflecting, even if the incident wave does not possess a geometrical shadow, there may be one or more geometrical shadows associated with the reflected waves.

Hence the solution for perfect reflexion is that found by Macdonald and Bromwich, starting with the theory of images.

It must be remembered, however, that the problems to which the method of images applies are isolated cases, and therefore to pass from them to diffraction, which has no connexion with perfect reflexion, is really an analytical convenience.

In this paper we have not considered the general problem of diffraction but we have endeavoured only to treat the wedge problem from a point of view which is free from certain objections.

Difficulties in the general problem of absorption have been pointed out, and even if we could deal with these difficulties the attempt would carry us beyond our present intentions.

Summary

Towards the end of last century and early in the present century the problem of the diffraction of waves by a wedge attracted considerable attention among mathematicians. The problem, in addition to its being attractive from the analytical standpoint, is important from the physical point of view, and an attempt at its solution by exact analysis naturally aroused attention as a means of verifying Fresnel's celebrated theory.

The problem was finally solved by Macdonald in 1915 with an extension by Bromwich. The former commences by obtaining Green's function for a wedge, while the latter takes the theory of images as his starting point. Some time previously Macdonald had propounded a theory of perfectly absorbing bodies, and when attempting to fit this theory to the wedge problem the author of the present paper was confronted with certain difficulties. Upon endeavouring to overcome these difficulties it was found that there are objections to basing the solution upon a perfectly conducting wedge as is done by Macdonald and Bromwich, and it appeared

to be a more fundamental and satisfactory process to base the solution upon the properties of the geometrical shadow.

This process has made it possible to obtain, by new and distinct methods, solutions both for the point source and the incident plane wave, and it has also made it possible to obtain the solution of a very interesting problem of diffraction in the presence of total reflexion at the surface of separation of two media of different refractive indices.

These are the main problems dealt with in the paper, while in the final section there is a discussion of the difficulties arising in MACDONALD's theory of absorbing bodies.

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